Critical State Mechanics in Non-Linear Failure Criterion for Rocks

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ABSTRACT

The strength of initially intact rock increases non-linearly with increase in the confining pressure. This paper proposes that there should be a saturation limit to the incremental gain in the frictional strength due to an increase in the confining pressure. Critical state of a rock is said to be reached if there is no further increase in deviator stress or shear stress at failure due to an increase in confining stress or normal stress. It is suggested that the coefficient of friction in terms of incremental shear strength/ normal stress is negligible beyond a critical confining stress, which is about uniaxial compressive strength of the rock material. The critical state should, therefore, be a part of the non-linear failure criterion. A simple parabolic failure criterion is suggested based on this hypothesis. The criterion involves only one unknown parameter compared to at least two in most of the other criteria in vogue, and may be obtained from a single triaxial test. The parameter may also be obtained from initial value of the coefficient of internal friction μ_o. As such the friction law is modified and a simple correlation is found to assess the coefficient of internal friction μ at any confining pressure. The modified friction law will improve our understanding of the non-linear geodynamics.

Keywords: Critical state rock mechanics, non-linear failure criterion, coefficient of internal friction

1. INTRODUCTION

At present our basic understanding of the strength of the earth's crust is based on a simple model that utilizes the friction law of rocks (Byerlee, 1978) in the shallower or 'brittle' layer of the crust and the plastic flow law in the deeper or 'ductile' layer (Goetze and Evans, 1979; Sibson, 1982; Meissner and Strehlau, 1982). Faulting and sliding on frictional surface is the primary mode of deformation in the upper lithosphere. A wide deformation conditions are possible and these must be investigated to comprehend earthquake source processes and factors that control lithospheric strength.

The Fig. 1 shows a simple model of interaction between colliding earth plates. Nedoma (1997) analysed the state of stresses in the earth plates, considering non-linear elasto-plastic –thermal behaviour of rocks and faults. The lower plate bends downwards, releasing horizontal tectonic stresses, which become minor principal stresses near the thrust. Thus there is subsidence and normal faulting in the (lower) subducting plate. On the contrary, the upper plate bends upwards, resulting in compression and higher tectonic stresses, which become major principal stresses. Hence this is the region of continuous uplifting (in the form of mountainous terrain) and thrust faulting near the inter-plate boundary. Simultaneously, its bottom part also bends upwards, releasing the tangential stresses. Two distressed (decompressed) zones are indicated by the stress analysis of Nedoma (1997). The temperature of melting of rocks is reduced by decrease in the confining stresses. The rock melts in these decompressed zones (Fig. 1). In nature, the molten rock may come out on the ground surface as a pair of volcanoes at some favourable geological situations. The critical state Rock Mechanics suggests that the coefficient of friction will tend to be nearly zero below the brittle crust.

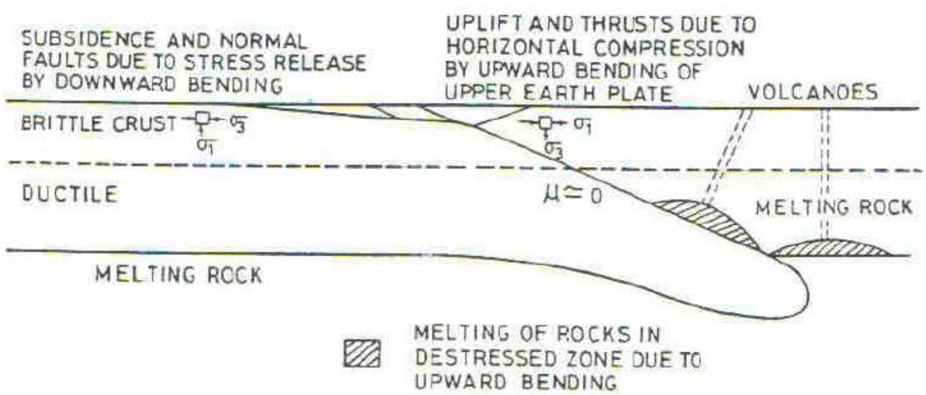

Fig. 1 - Interaction between boundaries of earth plates (Nedoma, 1997)

Coulomb's linear failure criterion has been used extensively to assess the shear strength of intact rocks. The criterion can be expressed as follows:

$$|\tau| = c + \mu_i \sigma_n \quad (1)$$

Where τ and σ_n are shear and normal effective stress resolved on the eventual fracture plane, c is cohesion and μ_i is the coefficient of internal friction (Lockner and Beeler, 2002). It is a well accepted fact now, that, linear approximation of the failure envelope can only be done if variation in σ_n is very small. For large variation, like those occurring at great depths in the earth crust, the non-linearity of the failure envelope plays a great role in assessing the available frictional strength of the rocks. Bilinear expressions have been suggested by Byerlee (1978) and continuously varying strength with increase in confining pressure has been suggested by Lockner (1998). The coefficient of friction μ, in this case, does not remain constant and varies with increase in the effective normal stress. For any ambient stress condition, the instantaneous coefficient of friction μ can be obtained as the gradient of the Mohr's failure surface i.e. $\mu = \partial\tau/\partial\sigma_n$ in τ, σ_n plane (Fig.2). It is proposed in this paper that the value of μ should approach zero as the rock reaches a critical state. Beyond this, there will be no more increase in the frictional strength due to an increase in confining stress or normal stress. The strength will therefore reach a saturation limit. The critical state in this paper is assumed when confining pressure equals the unconfined compressive strength of the rock. A strength criterion is proposed for initially intact rocks and friction law is modified.

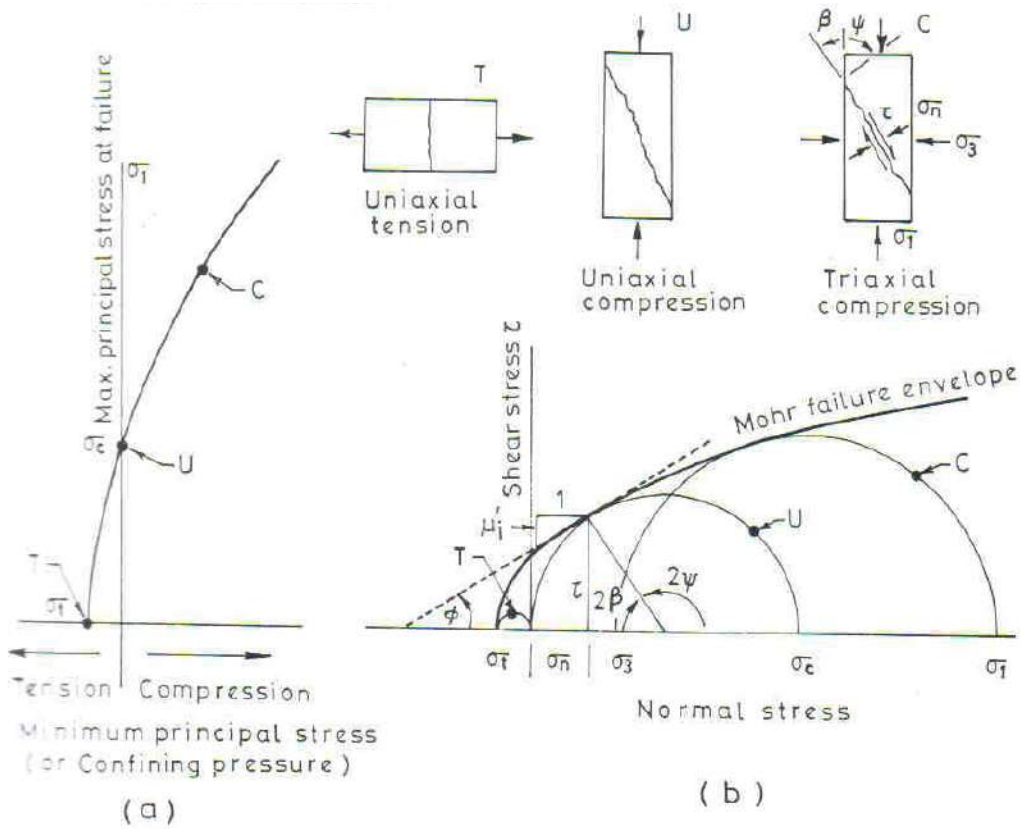

Fig. 2 - Mohr failure envelope showing relation between stresses and failure parameters (Lockner and Beeler, 2002)

The non-linear geodynamical analysis is performed in terms of the incremental shear strength ($\Delta\tau$) and incremental effective normal stress ($\Delta\sigma$) along an active fault with reference to the initial state of stress as follows:

$$|\Delta\tau| = \mu\,\Delta\sigma \tag{2}$$

The effect of cohesion in Eqn. (2) need not, therefore be considered. Evidently, $|\Delta\tau|$ cannot exceed the shear strength of adjoining weaker rock under high confining stress.

2. THE STRENGTH CRITERION

A strength criterion defines the failure surface. In present case a two dimensional form of the criterion is suggested in deviatoric stress (σ_1- σ_3) and σ_3 plane (Singh and Singh 2003); where σ_1 is the effective major principal stress at failure and σ_3 is the effective minor principal stress. The effect of the intermediate principal stress is not considered in the present form. A parabola is used to define the strength criterion as follows (Fig. 3).

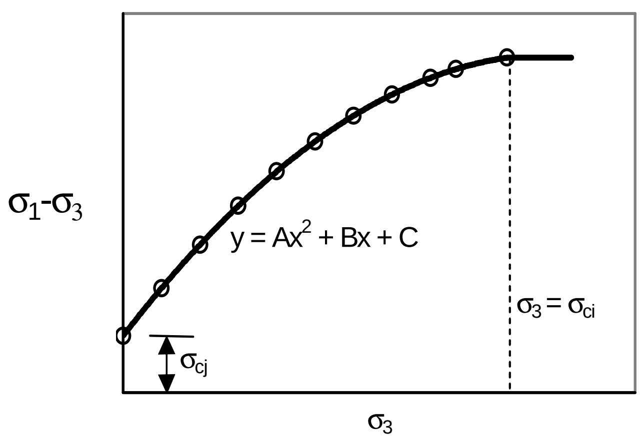

Fig. 3 - The proposed parabolic strength criterion

$$(\sigma_1 - \sigma_3) = A\,(\sigma_3)^2 + B\,(\sigma_3) + C \tag{3}$$

Substituting $\sigma_1 = \sigma_{ci}$ at $\sigma_3 = 0$; we get $C = \sigma_{ci}$; the criterion is written as:

$$(\sigma_1 - \sigma_3) = A\,(\sigma_3)^2 + B\,(\sigma_3) + \sigma_{ci} \tag{4}$$

Where σ_{ci} is the uniaxial compressive strength (UCS) of the rock material. To obtain parameter B, the critical state concept is applied. The critical state of initially intact rock is defined as the stress condition under which, Mohr envelope of peak shear strength reaches a point of zero gradient (Barton, 1976) as shown in Fig. 4. This represents the maximum possible shear strength of a rock and any further increment in confining stress or normal stress will not cause any increase in deviator stress or shear stress at failure. For critical state, there is an affective confining pressure above which, the shear strength cannot be made to increase (Barton, 1976). Also, the experimental evidence (Hoek, 1983) suggests that the Brittle-Ductile transition takes place at a confining pressure approximately equal to the unconfined compressive strength (UCS) of the rock material (Fig. 5). An approximate value of the critical confining pressure may, therefore, be taken equal to the UCS of the rock material. This value may however be modified with more experimental experience on a given rock material.

Differentiating Eq. (4)

$$\frac{\partial(\sigma_1-\sigma_3)}{\partial\sigma_3} = 2A\sigma_3 + B$$

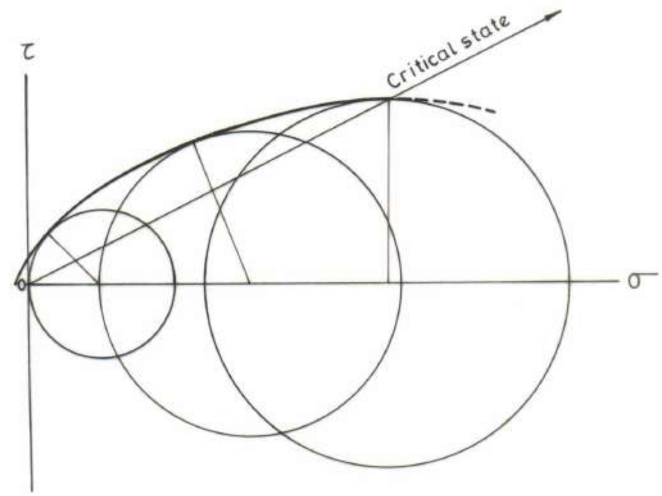

Fig. 4- Critical state of rock (Barton, 1976)

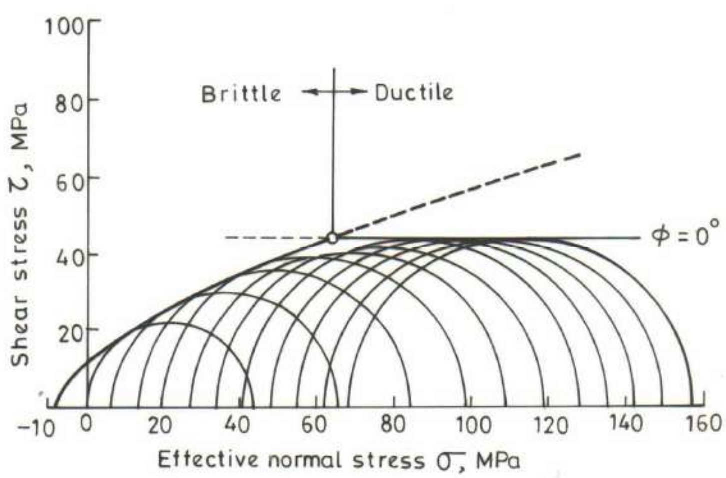

Fig. 5 - Brittle-Ductile transition (Hoek, 1983)

For critical state $\sigma_3 \rightarrow \sigma_{ci}$ and $\frac{\partial(\sigma_1 - \sigma_3)}{\partial \sigma_3} \rightarrow 0$

$$\Rightarrow B \;=\; -2\,A\,\sigma_{ci} \tag{5}$$

The strength criterion may therefore be written as

$$\sigma_1 = \begin{cases} A(\sigma_3)^2 + (1 - 2A\sigma_{ci})\sigma_3 + \sigma_{ci} & 0 < \sigma_3 \le \sigma_{ci} \\ 2\sigma_{ci} - A(\sigma_{ci})^2 & \sigma_3 \ge \sigma_{ci} \end{cases} \tag{6}$$

In addition to the uniaxial compressive strength of the rock, which is generally available for the concerned rock from laboratory tests, the above criterion has only one criterion parameter A.

The above criterion was applied to 132 data sets of triaxial test data of intact rocks, available from Sheorey (1997), and having uniaxial compressive strength varying from ≈7 to 500. The parameter A was computed for each rock by fitting experimental data through least square method and σ_1 value for each confining pressure was calculated. It is heartening to observe that the coefficient of correlation between the experimental and the calculated values of σ_1 is as high as 0.98 (Fig. 6). It proves the applicability of the strength criterion proposed and also the critical state of rocks at $\sigma_3 \approx \sigma_{ci}$. The non-linearity is due to mechanics of the critical state (or the law of saturation). It would be interesting to study the effect of critical state on seismic wave velocities in the rocks.

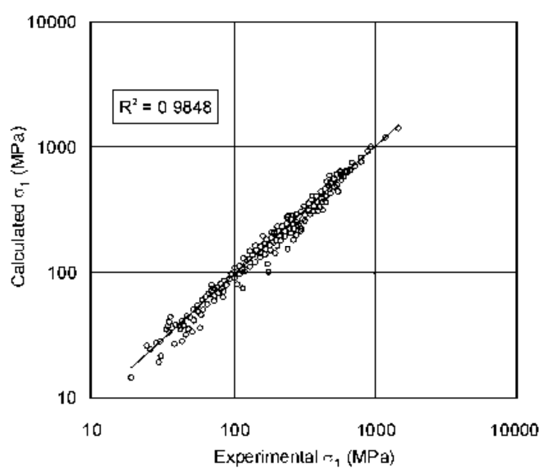

Fig.6 - Comparison of experimental major principal stress values with those calculated through the proposed criterion

Theoretically, only single triaxial test is required to assess the parameter A; however at least three tests are suggested for reliable assessment of the parameter. The parameter A, when plotted against the UCS of the rock, exhibits a strong correlation with the UCS (Fig. 7). In the absence of triaxial test results, a rough estimation of the parameter may be made through the correlation (Fig. 7) as follows (Singh and Singh, 2005):

$$A = -3.97\,(\sigma_{ci})^{-1.10} \qquad \text{for } \sigma_{ci} = 7 - 500 \text{ MPa} \tag{7}$$

$$\text{or} \quad B = 7.94\,(\sigma_{ci})^{-0.10} \tag{8}$$

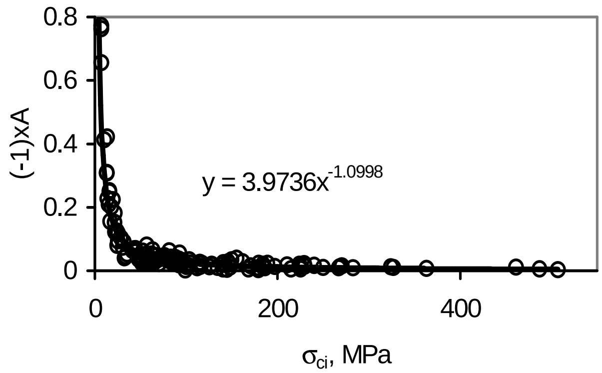

Fig. 7 - Variation of parameter A with UCS

3. FRICTIONAL RESISTANCE OF INTACT ROCKS

The fracture and failure of rocks is important in studies related to earthquakes (Lockner and Beeler, 2002) and plate tectonics (Shankar et al., 2000). The coefficient of internal friction is defined as the gradient of the failure surface i.e. $\mu = \partial\tau/\partial\sigma_n$ at given σ_n (Fig. 1). With increase in the depth of rocks in earthquake or plate tectonic studies, the overburden pressure increases. The assessment of the incremental shear strength through linear equation like Mohr-Coulomb criterion will overestimate the prediction of shear strength. Instead, a non-linear equation should be used to closely predict the shear strength within the brittle crust. The peak shear strength parameters c and ϕ should vary according to the level of normal stress and can be obtained by drawing tangent to the failure surface at the desired normal stress. Balmer (1952) has given expressions, using which, the instantaneous c and ϕ may be computed as follows:

$$\sigma_n = \sigma_3 + \frac{\sigma_1 - \sigma_3}{1 + \dfrac{\partial\sigma_1}{\partial\sigma_3}} \tag{9}$$

$$\tau_n = \frac{\sigma_1 - \sigma_3}{1 + \sqrt{\dfrac{\partial\sigma_1}{\partial\sigma_3}}} \sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}} \tag{10}$$

$$\tan\theta = \sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}} \tag{11}$$

$$\phi \;=\; 2\theta - 90^o, \qquad \mu = \tan^{-1}(2\theta - 90^o)$$

$$c \;=\; \tau_n - \sigma_n \tan\phi \tag{12}$$

where σ_n is the normal stress and τ_n is the corresponding shear stress on Mohr failure envelope.

3.1 Coefficient of Internal Friction under Unconfined State

The coefficient of internal friction μ_o, is easily available for different rock materials. It is shown here, that this value of μ_o may also be used to obtain the criterion parameters A and B; and the entire range of non-linear strength envelope may be generated.

Differentiating Eqn. (6)

$$\frac{\partial\sigma_1}{\partial\sigma_3} = 2A\,\sigma_3 + 1 - 2A\,\sigma_{ci} \quad \text{for } 0<\sigma_3\le\sigma_{ci} \tag{13}$$

Substituting $\sigma_3 \to 0$; $\sigma_1 \to \sigma_{ci}$

$$\left[\frac{\partial\sigma_1}{\partial\sigma_3}\right]_{\sigma_3\to 0} = 1 - 2\,A\,\sigma_{ci}$$

Also, substituting B for – 2 A σ_{ci}

$$\frac{\partial\sigma_1}{\partial\sigma_3} = 1 + B \tag{14}$$

Using Eqn. (11)

$$\tan^2\theta = 1 + B$$

$$\text{Putting } \theta = 45 + \frac{\phi}{2}$$

$$B = \frac{2\,\text{Sin}\,\phi}{1 - \text{Sin}\phi} \tag{15}$$

If, the coefficient of internal friction in unconfined state is denoted as μ_o, the parameter B and A may be obtained as

$$B = \frac{2\,\mu_o}{\sqrt{(1+\mu_o^2)} - \mu_o} \tag{16}$$

$$A = -\frac{\mu_o}{\sqrt{(1+\mu_o^2)}-\mu_o}\frac{1}{\sigma_{ci}} \tag{17}$$

Thus the advantage of the proposed criterion is that all the parameters A, B and C have conceptual meaning. It is applicable to weak rocks also (σ_{ci} >7 MPa).

3.2 Coefficient of Internal Friction under Confined State

The coefficient of internal friction μ, is not a constant value, rather it goes on reducing with the increase in effective confining pressure. An evidence of this is given by Shankar et al. (2000), who have back-analysed a friction angle (tan $\phi = \mu$) of only 5° beyond a depth of 40 km below the ground surface along the plate boundary in the Tibet Himalayan plate. It is interesting to note that lesser the frictional resistance along colliding inter-plate boundaries, lesser will be the locked up strain energy in the large earth plates and so lesser are the chances of great earthquakes in that area. In fact highest earthquake of only M7 on Richter's scale had taken place in the Tibetan pleatue. Thus there is balancing mechanism in the nature to avoid too high intensity of earthquakes.

To observe how μ varies with confining pressure, a rock having σ_{ci} = 50 MPa is assumed. A value of $\mu_{o,}$ say 0.7 is assumed and the parameters A and B are computed. Now σ_3 is varied from 0 to σ_{ci} and μ is computed for each confining pressure. The computations are repeated for several μ_o values. The variation of μ for different values of μ_o, is presented in Fig. 8, where, μ is plotted against non-dimensional parameter σ_3/σ_{ci}. It is interesting to see that if any other σ_{ci} is taken, the same plots are obtained. These unique plots can, therefore, be used for getting the μ value at any effective confining pressure. The following simple correlation, obtained by trial and error, has been found adequate to assess the μ for any initial μ_o (<2) and UCS (σ_{ci} >7 MPa).

$$\mu = \mu_o\left[1-\left(\frac{\sigma_3}{\sigma_{ci}}\right)^{\pi/2}\right] \tag{18}$$

$$\approx 0 \quad \text{for } \sigma_3 \geq \sigma_{ci}\ .$$

It may be seen in Fig. 9, that there is reasonable agreement between the values of μ that are derived from the Mohr's envelope and those from Eqn. 18. Further, the density of (hard) rocks is likely to increase slightly with the increasing confining stresses within the lithosphere. There may be a little gain in the strength after σ_3>σ_{ci}; as such μ in Eqn. 18 may be a small quantity and not zero for $\sigma_3 \geq \sigma_{ci}$. This hypothesis along with the effect of temperature needs further experimental studies for specific rock materials.

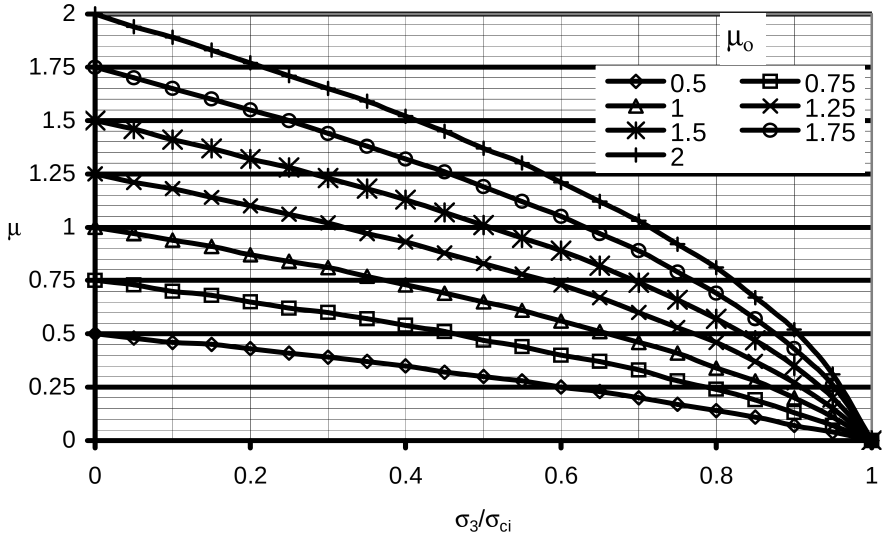

Fig 8 - Variation of coefficient of friction with confining pressure

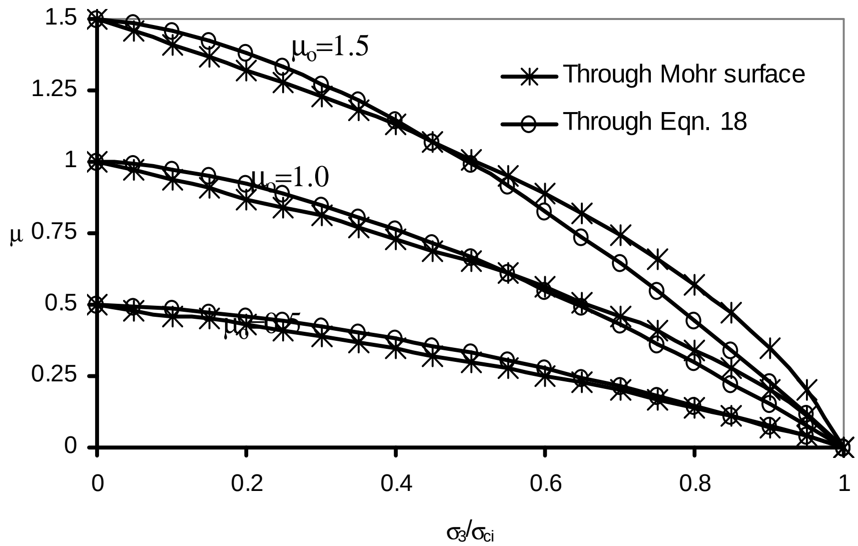

Fig. 9 - Comparison of coefficient of friction derived from Mohr's envelope with that obtained from Eq. 18

3.3 Maximum Shear Strength of a Rock

The critical state mechanics suggests that there is an upper limit to the deviator strength of rock materials. The maximum shear strength and corresponding normal stress of a rock may be obtained as follows:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1}{2}\left[A\left(\sigma_3\right)^2 \text{ - } 2A\,\sigma_{ci}\ \sigma_{c3} + \sigma_{ci}\right]$$

For maximum shear strength $\sigma_3 = \sigma_{ci}$

$$\tau_{max} = \frac{1}{2}\left[A\,\sigma_{ci}^2 \text{ - } 2A\,\sigma_{ci}^2 + \sigma_{ci}\right] = \frac{\sigma_{ci}}{2}\left[1 + \frac{B}{2}\right]$$

or

$$\tau_{max} = \frac{\sigma_{ci}}{2}\left[1 + \frac{\mu_o}{\sqrt{1-\mu_0{}^2} - \mu_0}\right] \tag{19}$$

The corresponding normal stress will be

$$\begin{aligned}(\sigma_n)_{max} &= \frac{(\sigma_1 - \sigma_3)_{max}}{2} + \sigma_3 \\ &= \left[1 + \frac{B}{2}\right]\frac{\sigma_{ci}}{2} + \sigma_{ci} \\ &= \frac{\sigma_{ci}}{2}\left[3 + \frac{B}{2}\right]\end{aligned}$$

or

$$(\sigma_n)_{max} = \frac{\sigma_{ci}}{2}\left[3 + \frac{\mu_o}{\sqrt{1+\mu_o^2} - \mu_0}\right] \tag{20}$$

Where τ_{max} and $(\sigma_n)_{max}$ are the maximum shear strength and corresponding effective normal stress on a fault plane in the rock. It is suggested that the incremental shear strength, in analysis of plate tectonics, may be taken nearly zero if the normal stress acting on the rock equals or exceeds the above critical value (Singh et al., 2004).

4. CONCLUDING REMARKS

It is well recognized that the failure envelope for rocks is not straight line but curvilinear and concave towards the normal stress axis. It is proposed in this paper that the critical state of rock should be a part of the non-linearity of the strength criterion. A simple parabolic strength criterion is proposed and the critical state is assumed at the confining pressure equal to the UCS of the rock material or as found experimentally. The coefficient of internal friction μ, which is the gradient of the failure surface, is observed to be varying non-linearly from a value of $\mu_{o,}$ in unconfined state, to about zero in the critical state. Plots have been presented to assess

μ at any given confining pressure (σ_3). The plots are drawn against the non-dimensional parameter (σ_3/σ_{ci}) and are found to be independent of the UCS of the rock, therefore may be used for all rock types in the non-linear geodynamical or any other analysis. A simple mathematical expression has also been suggested to compute μ accurately, at any confining pressure (Eq. 18).

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